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NOTE

On Alzer's Inequality

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A generalization of Alzer's inequality is proved. It is shown that this inequality is satisfied for a large class of increasing convex sequences. © 1998 Academic Press

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Alzer [1] proved the following inequality:

$$\frac{n}{n+1} \le \left((n+1) \sum_{i=1}^{n} i^r \Big/ n \sum_{i=1}^{n+1} i^r \right)^{1/r}, \tag{1}$$

where r is a positive real number and n is a natural number.

In this paper we shall prove (1) in a more general context, for a sequence (a_n) which satisfies an additional condition.

In [2, 3], an elementary proof for (1) is given. The proof in [3] is based on the following lemma:

LEMMA 1. *If r is a positive real number, then*

$$1 \le (1+w)^r \left[w + (1-w)^{r+1} \right] \qquad \textit{for } 0 \le w \le 1. \tag{2}$$

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We note that the proof of Lemma 1, given in [3], can also be simplified. Indeed, for each positive x_1, x_2, r, p_1, and p_2 $(p_1 + p_2 = 1)$,

$$\left(p_1 x_1^{r+1} + p_2 x_2^{r+1}\right)^{1/(r+1)} \geq p_1 x_1 + p_2 x_2 \tag{3}$$

(inequality between weighted means of order $r + 1$ and 1). It is now sufficient to take in (3) $p_1 = w/(w+1)$, $p_2 = 1/(w+1)$, $x_1 = 1$, and $x_2 = 1 - w$.

Instead of (1) we shall prove

$$\frac{a_n}{a_{n+1}} \leq \left(\frac{a_{n+1}\sum_{i=1}^{n} a_i^r}{a_n \sum_{i=1}^{n+1} a_i^r}\right)^{1/r}, \tag{4}$$

where r is a positive real number and (a_n), $n \geq 1$, is a sequence of positive real numbers.

THEOREM 1. *If the sequence* (a_n), $n \geq 1$, *of positive real numbers satisfies*

$$1 \leq \left(\frac{a_{n+2}}{a_{n+1}}\right)^r \left[\frac{a_{n+2}}{a_{n+1}} - 1 + \left(\frac{a_n}{a_{n+1}}\right)^{r+1}\right], \qquad n \geq 0,\ a_0 = 0, \tag{5}$$

then (4) *holds*.

Proof. The proof is by induction. For $n = 1$, (4) becomes

$$\frac{a_1^r}{a_2^r} \leq \frac{a_2 a_1^r}{a_1(a_1^r + a_2^r)},$$

which is equivalent to

$$1 \leq \left(\frac{a_2}{a_1}\right)^r \left(\frac{a_2}{a_1} - 1\right),$$

and this is exactly (5) for $n = 0$.

Let us prove the next step of induction. The inequality (4) is equivalent to

$$\sum_{k=1}^{n+1} a_k^r \geq \frac{a_{n+1}^{2r+1}}{a_{n+1}^{r+1} - a_n^{r+1}},$$

i.e.,

$$\sum_{k=1}^{n+2} a_k^r \geq \frac{a_{n+1}^{2r+1} + a_{n+2}^r a_{n+1}^{r+1} - a_{n+2}^r a_n^{r+1}}{a_{n+1}^{r+1} - a_n^{r+1}}.$$

Therefore, it suffices to prove

$$\frac{a_{n+1}^{2r+1} + a_{n+2}^{r} a_{n+1}^{r+1} - a_{n+2}^{r} a_{n}^{r+1}}{a_{n+1}^{r+1} - a_{n}^{r+1}} \geq \frac{a_{n+2}^{2r+1}}{a_{n+2}^{r+1} - a_{n+1}^{r+1}}. \tag{6}$$

But (6) is equivalent to (5).

COROLLARY 1. *Let the sequence* (a_n) *of positive real numbers satisfy*

$$\frac{a_2}{a_1} \geq \left(\frac{a_1}{a_2}\right)^r + 1, \tag{7}$$

$$a_n - 2a_{n+1} + a_{n+2} \geq 0, \qquad n \geq 1. \tag{8}$$

Then (4) *holds*.

Proof. Equation (7) is equivalent to (5) for $n = 0$. Let us denote $w = a_{n+2}/a_{n+1} - 1$. Then $w > 0$, since the convex sequence (a_n) which satisfies $a_2 > a_1$ must be increasing. If $a_{n+2} \leq 2a_{n+1}$ then $w \leq 1$ holds as well. But if $a_{n+2} \geq 2a_{n+1}$ then we have

$$\left(\frac{a_{n+2}}{a_{n+1}}\right)^r \left[\frac{a_{n+2}}{a_{n+1}} - 1 + \left(\frac{a_n}{a_{n+1}}\right)^{r+1}\right]$$
$$\geq \left(\frac{a_{n+2}}{a_{n+1}}\right)^r \left[\frac{a_{n+2}}{a_{n+1}} - 1\right] \geq \left(\frac{a_{n+2}}{a_{n+1}}\right)^r \geq 1.$$

So, we can suppose $0 < w \leq 1$. But, then, (8) implies

$$\frac{a_n}{a_{n+1}} \geq 1 - \left(\frac{a_{n+2}}{a_{n+1}} - 1\right)$$

and, hence, by Lemma 1,

$$\left(\frac{a_{n+2}}{a_{n+1}}\right)^r \left[\frac{a_{n+2}}{a_{n+1}} - 1 + \left(\frac{a_n}{a_{n+1}}\right)^{r+1}\right]$$
$$\geq \left(\frac{a_{n+2}}{a_{n+1}}\right)^r \left[\frac{a_{n+2}}{a_{n+1}} - 1 + \left\{1 - \left(\frac{a_{n+2}}{a_{n+1}} - 1\right)\right\}^{r+1}\right]$$
$$= (1 + w)^r \left[w + (1 - w)^{r+1}\right] \geq 1.$$

Thus, (a_n) satisfies (5), and the corollary follows.

COROLLARY 2. *For each strictly increasing convex sequence* (a_n) *of positive real numbers there exists an* $r > 0$ *such that* (4) *holds*.

Proof. The statement follows since (7) is satisfied for sufficiently large r.

EXAMPLE 1. The sequence $a_n = n$ satisfies (7) and (8). Hence, Theorem 1 generalizes Alzer's inequality.

EXAMPLE 2. The sequence $a_n = 2n - 1$ satisfies (7) and (8). Therefore, we have

$$\frac{2n-1}{2n+1} \le \left(\frac{(2n+1)\sum_{i=1}^{n}(2i-1)^r}{(2n-1)\sum_{i=1}^{n+1}(2i-1)^r} \right)^{1/r}.$$

EXAMPLE 3. The sequence $a_n = k(n-1) + 1$, $k > 0$, satisfies (8). Further, (7) is equivalent to

$$k(k+1)^r \ge 1. \tag{9}$$

Therefore, (4) holds for this sequence whenever (9) is valid.

EXAMPLE 4. The sequence $a_n = a^n$, $a > 1$, satisfies (8). Further, (7) is equivalent to

$$a \ge \frac{1}{a^r} + 1. \tag{10}$$

As in the previous example, for each $r > 0$ there exists $a > 1$ for which (10) is valid.

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